

ON THE EISENBUD-GREEN-HARRIS CONJECTURE

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ABSTRACT. It has been conjectured by Eisenbud, Green and Harris that if I is a homogeneous ideal in $k[x_1, \dots, x_n]$ containing a regular sequence f_1, \dots, f_n of degrees $\deg(f_i) = a_i$, where $2 \leq a_1 \leq \dots \leq a_n$, then there is a homogeneous ideal J containing $x_1^{a_1}, \dots, x_n^{a_n}$ with the same Hilbert function. In this paper we prove the Eisenbud-Green-Harris conjecture when f_i splits into linear factors for all i .

1. INTRODUCTION

Let $S = k[x_1, \dots, x_n]$ be a polynomial ring over a field k . The ring $S = \bigoplus_{d \geq 0} S_d$ is graded by $\deg(x_i) = 1$ for all i . In 1927, F. Macaulay proved that if $I = \bigoplus_{d \geq 0} I_d$ is a graded ideal in S , then there exists a lex ideal L such that L has the same Hilbert function as I [13]; i.e., every Hilbert function in S is attained by a lex ideal. Let M be a monomial ideal in S . It is natural to ask if we have the same result in S/M . In [5], Clements and Lindström proved that every Hilbert function in $S/\langle x_1^{a_1}, \dots, x_n^{a_n} \rangle$ is attained by a lex ideal, where $2 \leq a_1 \leq \dots \leq a_n$. In the case $a_1 = \dots = a_n = 2$, the result was obtained earlier by Katona [11] and Kruskal [?]. Another generalizations of Macaulay's theorem can be found in [17], [15] and [1].

Let f_1, \dots, f_n be a regular sequence in S such that $2 \leq a_1 = \deg(f_1) \leq \dots \leq a_n = \deg(f_n)$. A well known result says that $\langle f_1, \dots, f_n \rangle$ has the same Hilbert function as $\langle x_1^{a_1}, \dots, x_n^{a_n} \rangle$ (see Exercise 6.2. of [9]). It is natural to ask what happens if $I \subseteq S$ is a homogeneous ideal containing a regular sequence in fixed degrees. This question bring us to the Eisenbud-Green-Harris Conjecture, denoted by EGH.

Conjecture 1.1 (EGH [8]).

If I is a homogeneous ideal in S containing a regular sequence f_1, \dots, f_n of degrees $\deg(f_i) = a_i$, where $2 \leq a_1 \leq \dots \leq a_n$, then I has the same Hilbert function as an ideal containing $x_1^{a_1}, \dots, x_n^{a_n}$.

The original conjecture (see Conjecture 2.3) is equivalent to Conjecture 1.1 in the case $a_i = 2$ for all i (see Proposition 2.5). The EGH Conjecture is known to be true in few cases. The conjecture has been proven in the case $n = 2$ [16]. Caviglia and Maclagan [3] have proven that the EGH Conjecture is true if $a_j > \sum_{i=1}^{j-1} (a_i - 1)$ for all $j > 1$. Richert [16] says that the EGH Conjecture in degree 2 ($a_i = 2$ for all i) holds for $n \leq 5$, but this result was not published. Herzog and Popescu [10] proved that if k is a field of characteristic zero and I is minimally generated by generic quadratic forms, then the EGH Conjecture in degree 2 holds. Cooper [6, 7] has done some work in a geometric direction. She studies the EGH Conjecture for some cases with $n = 3$.

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Let f_1, \dots, f_n be a regular sequence in S such that f_i splits into linear factors for all i . For all $1 \leq i \leq n$, let $p_i \in S_1$ such that $p_i | f_i$. Since p_1, \dots, p_n must be a k -linear independent, it follows that the k -algebra map $\alpha : S \rightarrow S$ defined by $\alpha(x_i) = p_i$ for all $1 \leq i \leq n$, is a graded isomorphism. So the Hilbert function is preserved under this map and we may assume that $p_i = x_i$ for all i .

In Section 2, we give background information to the EGH Conjecture. In section 3, we study the dimension growth of some ideals containing a regular sequence $x_1 l_1, \dots, x_n l_n$, where $l_i \in S_1$ for all i . In section 4, we prove the EGH Conjecture when f_i splits into linear factors for all i . This answers a question of Chen, who asked if the EGH Conjecture holds when $f_i = x_i l_i$, where $l_i \in S_1$ for all $1 \leq i \leq n$ (see Example 3.8 of [4]).

2. BACKGROUND

A proper ideal I in S is called *graded* or *homogeneous* if it has a system of homogeneous generators. Let $R = S/I$, where I is a homogeneous ideal. The *Hilbert function* of I is the sequence $H(R) = \{H(R, t)\}_{t \geq 0}$, where

$$H(R, t) := \dim_k R_t = \dim_k S_t / I_t.$$

For simplicity, sometimes we denote the dimension of a k -vector space V by $|V|$ instead of $\dim_k V$. For a k -vector space $V \subseteq S_d$, where $d \geq 0$, we denote by $S_1 V$ the k -vector space spanned by $\{x_i v : 1 \leq i \leq n \wedge v \in V\}$. Throughout this paper $\mathbf{A} = (a_1, \dots, a_n) \in \mathbb{Z}^n$, where $2 \leq a_1 \leq \dots \leq a_n$. For a subset A of S , we denote by $\text{Mon}(A)$ the set of all monomials in A and let $A_u = \{j : x_j | u\}$, where $u \in \text{Mon}(S)$. The *support* of the polynomial $f = \sum_{u \in \text{Mon}(S)} a_u u$, where $a_u \in k$, is the set

$$\text{supp}(f) = \{u \in \text{Mon}(S) : a_u \neq 0\}.$$

A monomial $w \in S$ is called *square-free* if $x_i^2 \nmid w$, for all $1 \leq i \leq n$. We define the *lex order* on $\text{Mon}(S)$ by setting $\mathbf{x}^b = x_1^{b_1} \dots x_n^{b_n} <_{\text{lex}} x_1^{c_1} \dots x_n^{c_n} = \mathbf{x}^c$ if either $\deg(\mathbf{x}^b) < \deg(\mathbf{x}^c)$ or $\deg(\mathbf{x}^b) = \deg(\mathbf{x}^c)$ and $b_i < c_i$ for the first index i such that $b_i \neq c_i$. We recall the definitions of lex ideal and lex-plus-powers ideal.

Definition 2.1. • A graded ideal is called *monomial* if it has a system of monomial generators.

- A monomial ideal $I \subseteq S$ is called *lex*, if whenever $I \ni z <_{\text{lex}} w$, where w, z are monomials of the same degree, then $w \in I$.
- A monomial ideal I is *\mathbf{A} -lex-plus-powers* if there exists a lex ideal L such that $I = \langle x_1^{a_1}, \dots, x_n^{a_n} \rangle + L$.

Example 2.2. the ideal $I = \langle x_1^2, x_2^2, x_1 x_2 x_3, x_3^3 \rangle$ is a $(2, 2, 3)$ -lex-plus-powers ideal in $k[x_1, x_2, x_3]$, because $I = \langle x_1^2, x_2^2, x_3^3 \rangle + \langle x_1^3, x_1^2 x_2, x_1^2 x_3, x_1 x_2^2, x_1 x_2 x_3 \rangle$ and $\langle x_1^3, x_1^2 x_2, x_1^2 x_3, x_1 x_2^2, x_1 x_2 x_3 \rangle$ is a lex ideal in $k[x_1, x_2, x_3]$.

By Clements-Lindström's theorem, we obtain that for any graded ideal containing $\langle x_1^{a_1}, \dots, x_n^{a_n} \rangle$ there is a (a_1, \dots, a_n) -lex-plus-powers ideal with the same Hilbert function.

Let $p \geq 0$ and $\binom{s_q}{q} + \binom{s_{q-1}}{q-1} + \dots + \binom{s_1}{1}$ be the unique Macaulay expansion of p with respect to $q > 0$. Set $0^{(q)} = 0$ and $p^{(q)} = \binom{s_q}{q+1} + \binom{s_{q-1}}{q} + \dots + \binom{s_1}{2}$. In [8], Eisenbud, Green and Harris made the following conjecture.

Conjecture 2.3. If $I \subset S$ is a graded ideal such that I_2 contains a regular sequence of maximal length and $d > 0$, then $H(S/I, d+1) \leq H(S/I, d)^{(d)}$.

Conjecture 2.3 is true if the ideal contains the squares of the variables. This follows from the Kruskal-Katona theorem (see [2]). In the following proposition, we prove the equivalence of Conjecture 2.3 and the EGH Conjecture in degree 2. First, we need the following definition.

Definition 2.4. Let M be a monomial ideal in S and $d \geq 0$. A monomial vector space L_d in $(S/M)_d$ is called *lexsegment* if it is generated by the t biggest monomials (with respect to the lex order) in $(S/M)_d = S_d/M_d$, for some $t \geq 0$.

For example, if L is a lex ideal in S , then L_j is lexsegment for all $j \geq 0$. If L_d is a lexsegment space in $(S/M)_d$, where M is a monomial ideal in S , then $S_1 L_d$ is lexsegment in $(S/M)_{d+1}$ (see Proposition 2.5 of [15]).

Proposition 2.5. Let f_1, \dots, f_n be a regular sequence of degrees 2 in S . The following are equivalent:

- (a) If I is a graded ideal in S containing f_1, \dots, f_n , then there is a graded ideal J in S containing x_1^2, \dots, x_n^2 such that $H(S/I) = H(S/J)$.
- (b) If I is a graded ideal in S containing f_1, \dots, f_n , then

$$H(S/I, d+1) \leq H(S/I, d)^{(d)} \text{ for all } d > 0.$$

Proof. First, we prove that (a) implies (b). Let I be a graded ideal in S containing f_1, \dots, f_n . By (a), it follows that there is a graded ideal J in S containing x_1^2, \dots, x_n^2 such that $H(S/I) = H(S/J)$. By Kruskal-Katona theorem it follows that $H(S/I, d+1) = H(S/J, d+1) \leq H(S/J, d)^{(d)} = H(S/I, d)^{(d)}$ for all $d > 0$.

Now, we prove that (b) implies (a). Let I be a graded ideal in S containing f_1, \dots, f_n . Set $M = \langle x_1^2, \dots, x_n^2 \rangle$ and $P = \langle f_1, \dots, f_n \rangle$. For every $d \geq 0$, let L_d be the k -vector space spanned by the first square-free monomials (in lex order) of S_d such that $|L_d \oplus M_d| = |I_d|$. Let $K = \bigoplus_{j \geq 0} K_j = \bigoplus_{j \geq 0} (L_j + M_j)$. We need to show that K is an ideal. Let $d > 0$. By Proposition 6.4.3 of [9], we obtain that

$$|S_{d+1}/M_{d+1}| - |S_1 L_d / S_1 L_d \cap M_{d+1}| = (|S_d/M_d| - |L_d|)^{(d)}.$$

By the hypothesis of (b), we obtain $(|S_d/M_d| - |L_d|)^{(d)} = |S_d/I_d|^{(d)} \geq |S_{d+1}/I_{d+1}|$. So $|S_{d+1}/M_{d+1}| - |S_1 L_d / S_1 L_d \cap M_{d+1}| = |S_{d+1}/M_{d+1}| - |S_1 L_d + M_{d+1}/M_{d+1}| \geq |S_{d+1}/I_{d+1}|$.

This implies that $|S_1 L_d + M_{d+1}| \leq |L_{d+1} + M_{d+1}|$. Since $\overline{L_{d+1}}$ and $\overline{S_1 L_d}$ are lexsegments in $(S/M)_{d+1}$, it follows that $S_1 L_d \subseteq L_{d+1} + M_{d+1}$. So $S_1 K_d \subseteq K_{d+1}$ for all $d \geq 0$, which implies that K is a graded ideal in S . Clearly, $H(S/K) = H(S/I)$. \square

The following lemma helps us to study the EGH Conjecture in each component of the homogeneous ideal.

Lemma 2.6. Let I be a graded ideal in S containing a regular sequence f_1, \dots, f_n of degrees $\deg(f_i) = a_i$. The following are equivalent:

- (a) There exists a graded ideal J in S containing $x_1^{a_1}, \dots, x_n^{a_n}$ such that $H(S/I) = H(S/J)$.
- (b) For every $d \geq 0$, there exists a graded ideal J in S containing $x_1^{a_1}, \dots, x_n^{a_n}$ such that $H(S/I, d) = H(S/J, d)$ and $H(S/I, d+1) \leq H(S/J, d+1)$.

Proof. Clearly, (a) implies (b). We will show that (b) implies (a). For every $d \geq 0$, there exists an ideal J_d in S containing $x_1^{a_1}, \dots, x_n^{a_n}$ such that $H(S/I, d) = H(S/J_d, d)$ and $H(S/I, d+1) \leq H(S/J_d, d+1)$. By Clements-Lindström's theorem, we may assume that J_d is a \mathbf{A} -lex-plus-powers ideal for all d . Let $J = \bigoplus_{j \geq 0} J_{j,j}$, where

$J_{j,j}$ is the j -th component of J_j . Since $\dim(J_{d,d+1}) \leq \dim(I_{d+1}) = \dim(J_{d+1,d+1})$, it follows that $J_{d,d+1} \subseteq J_{d+1,d+1}$, for all d . So $S_1 J_{d,d} \subseteq J_{d,d+1} \subseteq J_{d+1,d+1}$, for all d . Thus, J is an ideal. Clearly, $H(S/I) = H(S/J)$. \square

We will use the following lemma on regular sequences (see [14, Chapter 6]).

Lemma 2.7. *Let f_1, \dots, f_n be a sequence of homogeneous polynomials in S with $\deg(f_i) = a_i$ and $P = \langle f_1, \dots, f_n \rangle$. Then*

- (a) *If f_1, \dots, f_n is a regular sequence, then $H(S/P) = H(S/\langle x_1^{a_1}, \dots, x_n^{a_n} \rangle)$.*
- (b) *f_1, \dots, f_n is a regular sequence if and only if the following condition holds: if $g_1 f_1 + \dots + g_n f_n = 0$ for some $g_1, \dots, g_n \in S$, then $g_1, \dots, g_n \in P$.*
- (c) *If f_1, \dots, f_n is a regular sequence and $\sigma \in S_n$ is a permutation, then $f_{\sigma(1)}, \dots, f_{\sigma(n)}$ is a regular sequence.*

3. THE DIMENSION GROWTH OF SOME IDEALS CONTAINING A REDUCIBLE REGULAR SEQUENCE

Let $f_1 = x_1 l_1, \dots, f_n = x_n l_n$ be a regular sequence in S , where $l_i \in S_1$ for all i . Set $P = \langle f_1, \dots, f_n \rangle$ and $M = \langle x_1^2, \dots, x_n^2 \rangle$. Let V_d be a vector space spanned by P_d and square-free monomials w_1, \dots, w_t in S_d , and W_d be the vector space spanned by M_d and w_1, \dots, w_t . In this section, we prove that $\dim(S_1 V_d) = \dim(S_1 W_d)$. We also compute $\dim(S_1 K_d)$, where K_d is the space generated by P_d and the biggest (in lex order) square-free monomials v_1, \dots, v_t in S_d .

For a matrix $A \in M_{n \times n}(k)$, we denote by $A[i_1, \dots, i_r]$ the submatrix of A formed by rows i_1, \dots, i_r and columns i_1, \dots, i_r , where $1 \leq r \leq n$ and $1 \leq i_1 < \dots < i_r \leq n$. We begin with the following lemma, which characterizes the structure of f_1, \dots, f_n .

Lemma 3.1. (Example 3.8 of [4])

Let $f_1 = x_1 l_1, \dots, f_n = x_n l_n$ be a sequence of homogeneous polynomials in S , where $l_i = \sum_{j=1}^n a_{ij} x_j$ with $a_{ij} \in k$ and A be the $n \times n$ matrix (a_{ij}) . Then f_1, \dots, f_n is a regular sequence if and only if $\det A[i_1, \dots, i_r] \neq 0$ for all $1 \leq r \leq n$ and $1 \leq i_1 < \dots < i_r \leq n$.

Proof. Assume that f_1, \dots, f_n is regular. We prove that $\det A[i_1, \dots, i_r] \neq 0$ for all $1 \leq r \leq n$ and $1 \leq i_1 < \dots < i_r \leq n$, by induction on n , starting with $n = 1$. Let $n > 1$. Assume that $1 \leq i_1 < \dots < i_r \leq n$, where $1 \leq r \leq n - 1$. Let $j \notin \{i_1, \dots, i_r\}$. Note that $x_j l_j$ is regular modulo an ideal I if and only if both x_j and l_j are regular modulo I . By Lemma 2.7, $x_j, f_1, \dots, f_{j-1}, f_{j+1}, \dots, f_n$ is a regular sequence. So $\overline{f_1}, \dots, \overline{f_{j-1}}, \overline{f_{j+1}}, \dots, \overline{f_n}$ is a regular sequence in $S/\langle x_j \rangle$. By the inductive step we obtain that $\det A[i_1, \dots, i_r] \neq 0$. It remains to show that $\det(A) \neq 0$. From the permutability property of regular sequences of homogeneous polynomials, we obtain that l_1, \dots, l_n is a regular sequence. So l_1, \dots, l_n is k -linearly independent.

Assume now $\det A[i_1, \dots, i_r] \neq 0$ for all $1 \leq r \leq n$ and $1 \leq i_1 < \dots < i_r \leq n$. We prove that f_1, \dots, f_n is a regular sequence by induction on n , starting with $n = 1$. Let $n > 1$. By the inductive step, the sequence $\overline{f_1}, \dots, \overline{f_{n-1}}$ is regular in $S/\langle x_n \rangle$. So f_1, \dots, f_{n-1}, x_n is a regular sequence in S . It remains to show that f_1, \dots, f_{n-1}, l_n is a regular sequence. Since $\det(A) \neq 0$, it follows that the k -algebra map $\alpha: S \rightarrow S$ defined by $\alpha(x_i) = l_i$, for all i , is an isomorphism. By the inductive step, $\alpha^{-1}(f_1), \dots, \alpha^{-1}(f_{n-1}), \alpha^{-1}(l_n) = x_n$ is a regular sequence. So f_1, \dots, f_{n-1}, l_n is a regular sequence, as desired. \square

The special structure of the regular sequence in 3.1 implies the following lemma.

Lemma 3.2. *Let $f_1 = x_1 l_1, \dots, f_n = x_n l_n$ be a regular sequence of homogeneous polynomials in S , where $l_i = \sum_{j=1}^n a_{ij} x_j$ with $a_{ij} \in k$, and $P = \langle f_1, \dots, f_n \rangle$. If $g \notin P$ is a homogeneous polynomial in S , then*

$$g \equiv h \pmod{P}$$

where $\deg(h) = \deg(g)$ and h is a k -linear combination of square-free monomials.

Proof. Since $g \notin P$, we have $\deg(g) \leq n$. It is sufficient to prove the lemma when $g \notin P$ is a monomial in $\langle x_1^2, \dots, x_n^2 \rangle$ of degree $\leq n$. We prove by induction on $\deg(g)$. The lemma is true when $\deg(g) = 2$, since $a_{ii} \neq 0$ for all i . Let g be a monomial in $\langle x_1^2, \dots, x_n^2 \rangle$ of degree $d > 2$ and A be the $n \times n$ matrix (a_{ij}) . By the inductive step, we may assume that $\frac{g}{x_i}$ is a square-free monomial for some i . By Lemma 3.1, we have $\det A[j : j \in A_g] \neq 0$. So there exist scalars $(c_j)_{j \in A_g}$, such that $\sum_{j \in A_g} c_j l_j \equiv x_i \pmod{\langle x_j : j \notin A_g \rangle}$. It follows that $x_i = \sum_{j \in A_g} c_j l_j + \sum_{j \notin A_g} c_j x_j$, where $c_j \in k$ for all $j \notin A_g$. Then $g = \sum_{j \in A_g} c_j l_j \frac{g}{x_i} + \sum_{j \notin A_g} c_j x_j \frac{g}{x_i}$. Let $h = \sum_{j \notin A_g} c_j x_j \frac{g}{x_i}$. Note that $h \neq 0$ is a k -linear combination of square-free monomials of degree d . Since $\sum_{j \in A_g} c_j l_j \frac{g}{x_i} \in P$, we obtain that $g \equiv h \pmod{P}$. \square

By the proof of Lemma 3.2, we obtain the following.

Remark 3.3. Let P be as in Lemma 3.2 and $0 \leq d \leq n$. If w is a square-free monomial in S_d and $q \in S_1$, then

$$qw = \tilde{q}w + \hat{q}w$$

where $\tilde{q}, \hat{q} \in S_1$, $\hat{q}w \in P$ and $\tilde{q}w$ is a k -linear combination of square-free monomials.

Example 3.4. Assume that $S = \mathbb{C}[x_1, x_2, x_3]$ and

$$f_1 = x_1^2 + x_1 x_2 + x_1 x_3 = x_1(x_1 + x_2 + x_3)$$

$$f_2 = -x_1 x_2 + x_2^2 + x_2 x_3 = x_2(-x_1 + x_2 + x_3)$$

$$f_3 = -x_1 x_3 - x_2 x_3 + x_3^2 = x_3(-x_1 - x_2 + x_3).$$

In this case, $A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ -1 & -1 & 1 \end{pmatrix}$ is the matrix that defined in Lemma 3.1.

Since $\det A[i_1, \dots, i_r] \neq 0$ for all $1 \leq r \leq 3$ and $1 \leq i_1 < \dots < i_r \leq 3$, we have that f_1, f_2, f_3 is a regular sequence in S . Set $P = \langle f_1, f_2, f_3 \rangle$ and let $g = x_1^3 + x_1^2 x_2$. Since $x_1^2 \equiv -x_1 x_2 - x_1 x_3 \pmod{P}$, we have $x_1^3 \equiv -x_1^2 x_2 - x_1^2 x_3 \pmod{P}$. So $g \equiv -x_1^2 x_3 \pmod{P}$. Also, we see that $x_3 f_1 - x_1 f_3 = 2x_1^2 x_3 + 2x_1 x_2 x_3 \in P$. So $g \equiv x_1 x_2 x_3 \pmod{P}$ and $x_1 x_2 x_3 \notin \langle x_1^2, x_2^2, x_3^2 \rangle$.

Remark 3.5. Lemma 3.2 is not true if f_1, \dots, f_n is an arbitrary regular sequence. For example, consider the sequence

$$f_1 = x_1^2 + x_1 x_2 + x_2^2, \quad f_2 = x_1 x_2 \text{ in } \mathbb{C}[x_1, x_2].$$

Note that f_1, f_2 is a regular sequence $\Leftrightarrow f_1, x_1$ and f_1, x_2 are regular sequences $\Leftrightarrow x_1, f_1$ and x_2, f_1 are regular sequences $\Leftrightarrow x_2^2$ and x_1^2 are regular elements in $\mathbb{C}[x_2]$ and $\mathbb{C}[x_1]$, respectively. So f_1, f_2 is a regular sequence. Let $g = x_2^2$. It is easy to show that $g \notin \langle f_1, f_2 \rangle$. If $g \equiv a x_1 x_2 \pmod{\langle f_1, f_2 \rangle}$, for some $a \in \mathbb{C}$, then there exist $c_1, c_2, c_3 \in \mathbb{C}$, not all zero, such that $c_1 f_1 + c_2 f_2 + c_3 (g - a x_1 x_2) = 0$. But the equation

$$c_1 x_1^2 + (c_1 + c_2 - a c_3) x_1 x_2 + (c_1 + c_3) x_2^2 = 0,$$

implies that $c_1 = c_2 = c_3 = 0$, a contradiction.

As a result of Lemma 3.2, we obtain the following.

Lemma 3.6. *If P as in Lemma 3.2, then the set of all square-free monomials form a k -basis of S/P .*

Proof. Denote by \mathcal{A} the set of all square-free monomials in S . Lemma 3.2 shows that S/P generated by \mathcal{A} . Let $w = x_1 \cdots x_n$. Assume that $w \in P$. Since $H(S/P) = H(S/\langle x_1^2, \dots, x_n^2 \rangle)$, it follows that there is a polynomial $f \in S_n$ such that $f \notin P$. By Lemma 3.2, $f \equiv bx_1 \cdots x_n \pmod{P}$, where $0 \neq b \in k$. Since $w \in P$, it follows that $f \in P$, a contradiction. So $w \notin P$. Suppose that $\sum_{w \in \mathcal{A}} a_w w \in P$, where $a_w \in k$ and $a_w = 0$ for almost all $w \in \mathcal{A}$. Assume that $a_w \neq 0$ for some w . Let $v \in \mathcal{A}$ be a monomial with minimal degree such that $a_v \neq 0$. So $\overline{v} \in \langle \overline{f_i} : i \in A_v \rangle$ in the ring $S/\langle x_i : i \in A_v \rangle$, a contradiction. \square

Lemma 3.7. *Let P be as in Lemma 3.2. If w is a square-free monomial in S_d , where $0 \leq d \leq n$, then*

- (a) $|S_1(w) \cap P_{d+1}| = d$.
- (b) $|S_1(w) \cap (P_{d+1} + S_1(w_1, \dots, w_t))| = |S_1(w) \cap P_{d+1}| + |S_1(w) \cap S_1(w_1, \dots, w_t)|$
for every square-free monomials w_1, \dots, w_t of degrees d such that $w_i \neq w$ for all $1 \leq i \leq t$.

Proof. (a). Let $q = \sum_{i=1}^n c_i l_i \in S_1$, where $c_i \in k$ for all i , such that $qw \in P_{d+1}$. Assume that $c_j \neq 0$ for some $j \notin A_w$. Since $qw \prod_{j \notin A_w} x_j \in P$, it follows that $c_j l_j w \prod_{j \notin A_w} x_j \in P$. Thus, $c_j l_j w \prod_{j \notin A_w} x_j = h_1 f_1 + \dots + h_n f_n$, where $h_i \in S$ for all $1 \leq i \leq n$. So

$$h_1 f_1 + \dots + h_{j-1} f_{j-1} + (x_j h_j - c_j w \prod_{j \notin A_w} x_k) l_j + h_{j+1} f_{j+1} + \dots + h_n f_n = 0,$$

which implies that

$$x_j h_j - c_j w \prod_{j \notin A_w} x_k \in \langle f_1, \dots, f_{j-1}, f_{j+1}, \dots, f_n \rangle.$$

So $\overline{w \prod_{j \notin A_w} x_k} \in \langle \overline{f_1}, \dots, \overline{f_{j-1}}, \overline{f_{j+1}}, \dots, \overline{f_n} \rangle$ in the ring $S/\langle x_j \rangle$, a contradiction to Lemma 3.6. It follows that q belong to the k -vector space $(l_i : i \in A_w)$. On the other hand, $l_i w \in P$, for all $i \in A_w$. So $|S_1(w) \cap P_{d+1}| = \dim(l_i w : i \in A_w) = d$.

(b). First, we show that

$$S_1(w) \cap (P_{d+1} + S_1(w_1, \dots, w_t)) = S_1(w) \cap P_{d+1} + S_1(w) \cap S_1(w_1, \dots, w_t).$$

Assume that $qw \in P_{d+1} + S_1(w_1, \dots, w_t)$, where $q \in S_1$. There exist $f \in S_1(w_1, \dots, w_t)$ and $g \in P_{d+1}$ such that $qw = g + f$. If $f \in P$, then $qw \in S_1(w) \cap P_{d+1}$. So assume that $f \notin P$. By 3.3, we may assume that f is a k -linear combination of square-free monomials. Also, we obtain that $qw = \tilde{q}w + \hat{q}w$, where $\tilde{q}, \hat{q} \in S_1$, $\hat{q}w \in P$ and $\tilde{q}w$ is a k -linear combination of square-free monomials. So $\tilde{q}w - f \in P$, which implies that $\tilde{q}w = f \in S_1(w_1, \dots, w_t)$. Hence $qw \in S_1(w) \cap P_{d+1} + S_1(w) \cap S_1(w_1, \dots, w_t)$ and we obtain that the desired equality.

It remains to show that

$$S_1(w) \cap S_1(w_1, \dots, w_t) \cap P_{d+1} = (0).$$

Let $qw \in S_1(w_1, \dots, w_t) \cap P_{d+1}$, where $q \in S_1$. By (a), we have $q = \sum_{j \in A_w} c_j l_j$, where $c_j \in k$ for all $j \in A_w$. For every $1 \leq j \leq t$, let $i_j \in A_{w_j} \setminus A_w$ and let $B = \{i_j : 1 \leq j \leq t\}$. By the hypothesis, we obtain that $qw = \sum_{i=1}^t q_i w_i$, where $q_i \in S_1$ for all $1 \leq i \leq t$. So

$\overline{qw} = \overline{0}$ in the ring $S/\langle x_j : j \in B \rangle$, which implies that $\sum_{j \in A_w} \overline{c_j l_j} = \overline{0}$. By 3.1, we obtain that $c_j = 0$, for all $j \in A_w$. Thus, $qw = 0$. \square

Remark 3.8. Part (b) of Lemma 3.7 is not true if we replace w, w_1, \dots, w_t by homogeneous polynomials which are a k -linear combination of square-free monomials in S_d . For example, let $S = k[x_1, x_2, x_3, x_4]$ and $P = \langle x_1^2, x_2^2, x_3^2, x_4^2 \rangle$. Suppose that $h = x_1x_2 + x_2x_4 + x_3x_4$ and $h_1 = x_1x_2 + x_1x_3$. Computation with Macaulay2 shows that

$$|S_1(h) \cap (P_3 + S_1(h_1))| = 2 \text{ and } |S_1(h) \cap P_3| = |S_1(h) \cap S_1(h_1)| = 0.$$

In the case that w is a homogeneous polynomial in part (a) of Lemma 3.7, the dimension is always bounded by the degree. This is a result of the following proposition.

Proposition 3.9. *Let P be as in Lemma 3.2. If $g \notin P$ is a homogeneous polynomial of degree d , then $|S_1(g) \cap P_{d+1}| \leq d$.*

Proof. We prove by induction on n . If $n = 1$, then $g = ax_1$ or $g \in k$, where $a \in k$. If $g \in k$, then $|S_1(g) \cap P_1| = 0$ and if $g = ax_1$, then $|S_1(g) \cap P_2| = 1$. Let $n > 1$. We prove by induction on d , starting with $d = 0$. Let $d > 0$. If $d = n$, then $P_{d+1} = S_{d+1}$ and so $|S_1(g) \cap P_{d+1}| = n$. Assume that $d < n$. By 3.2, there exists a k -linear combination of square-free monomials $h \in S_d$ such that $g \equiv h \pmod{P_d}$. Clearly, $S_1(h) \cap P_{d+1} = S_1(g) \cap P_{d+1}$. Let $h = \sum_{i=1}^t a_i w_i$, where $0 \neq a_i \in k$ and $w_i \in \text{Mon}(S_d)$ for all i . Let $j \notin A_{w_1}$. If $l_j h \in P_{d+1}$, then $l_j w_1 \in \overline{P_{d+1}}$ in the ring $S/\langle x_i : i \notin A_w \wedge i \neq j \rangle$, a contradiction. So $l_j h \notin P_{d+1}$ for all $j \notin A_{w_1}$. In particular, there exists a variable x_i such that $x_i h \notin P_{d+1}$. We have two cases:

Case 1. $\overline{h} \notin \overline{P_d}$ in the ring $S/\langle x_i \rangle$. Let $\overline{p_1 h}, \dots, \overline{p_s h}$ be a basis of $\overline{S_1(h)} \cap \overline{P_{d+1}}$ in the ring $S/\langle x_i \rangle$. By the inductive step, we obtain that $s \leq d$. If $f \in S_1(h) \cap P_{d+1}$, then $f \in (p_1 h, \dots, p_s h, x_i q)$, where $q \in S_d$. Since $f \in S_1(h)$, it follows that $x_i q = r h$, where $r \in S_1$. Since $x_i \nmid h$, it follows that $x_i | r$. So $f \in (p_1 h, \dots, p_s h, x_i h)$. Therefore, $S_1(h) \cap P_{d+1} \subseteq (p_1 h, \dots, p_s h, x_i h)$. If $|S_1(h) \cap P_{d+1}| = s + 1$, then $x_i h \in P_{d+1}$, a contradiction.

Case 2. $\overline{h} \in \overline{P_d}$ in the ring $S/\langle x_i \rangle$. So $h \equiv x_i q \pmod{P_d}$, where $q \in S_{d-1}$. Since h is the unique k -linear combination of square-free monomials such that $x_i q \equiv h \pmod{P}$, we obtain that $h = x_i h_1$, where $h_1 \in S_{d-1}$. If $f \in S_1(h) \cap P_{d+1}$, then $f = p x_i h_1$, for some $p \in S_1$. Clearly, $\frac{f}{x_i} \in S_1(h_1)$. Since $f \in P$, it follows that $\overline{p h_1} \in \overline{P_d}$ in the ring $S/\langle l_i \rangle$. So $\frac{f}{x_i} \in \overline{S_1(h_1)} \cap \overline{P_d}$ in $S/\langle l_i \rangle$. If $\overline{h_1} \in \overline{P_{d-1}}$ in $S/\langle l_i \rangle$, then $x_i h_1 = h \in P$, a contradiction. Let $\overline{p_1 h_1}, \dots, \overline{p_s h_1}$ be a basis of $\overline{S_1(h_1)} \cap \overline{P_d}$. By the inductive step, we obtain that $s \leq d - 1$. So $\frac{f}{x_i} \in (p_1 h_1, \dots, p_s h_1, l_i q)$, which implies that $f \in (p_1 h, \dots, p_s h, l_i x_i q)$. Therefore, $|S_1(h) \cap P_{d+1}| \leq s + 1 \leq d$. \square

Now, we prove the main results of this section.

Theorem 3.10. *Let P be as in Lemma 3.2 and $M = \langle x_1^2, \dots, x_n^2 \rangle$. Assume that $V = P_d + (w_1, \dots, w_t)$ and $W = M_d + (w_1, \dots, w_t)$, where w_i is a square-free monomial of degree d , for all i . Then*

$$\dim S_1 W = \dim S_1 V.$$

Proof. We may assume that $d \geq 2$ and prove by induction on t . If $t = 1$, then

$$\begin{aligned} \dim S_1 W &= \dim M_{d+1} + \dim S_1(w_1) - \dim S_1(w_1) \cap M_{d+1} \\ &= \dim P_{d+1} + \dim S_1(w_1) - \dim S_1(w_1) \cap P_{d+1} = \dim S_1 V. \end{aligned}$$

Let $t > 1$, and set $W_1 = M_d + (w_1, \dots, w_{t-1})$, $V_1 = P_d + (w_1, \dots, w_{t-1})$ and $Z = S_1(w_t) \cap S_1(w_1, \dots, w_{t-1})$. By Lemma (3.7) and the inductive step, we have

$$\begin{aligned} \dim S_1 W &= \dim S_1 W_1 + \dim S_1(w_t) - \dim S_1(w_t) \cap S_1 W_1 \\ &= \dim S_1 V_1 + \dim S_1(w_t) - \dim S_1(w_t) \cap S_1 W_1 \\ &= \dim S_1 V_1 + \dim S_1(w_t) - \dim S_1(w_t) \cap M_{d+1} - \dim Z \\ &= \dim S_1 V_1 + \dim S_1(w_t) - \dim S_1(w_t) \cap P_{d+1} - \dim Z \\ &= \dim S_1 V_1 + \dim S_1(w_t) - \dim S_1(w_t) \cap S_1 V_1 \\ &= \dim S_1 V. \end{aligned}$$

□

Proposition 3.11. *Let P be as in Lemma 3.2 and $V = P_d + (w_1, \dots, w_t)$ be the k -vector space spanned by P_d and the t biggest (in lex order) square-free monomials in S_d . Then*

$$\dim S_1 V = \binom{d+n}{d+1} - \binom{n}{d+1} + \sum_{i=1}^t (n - m(w_i)),$$

where $m(w_i) = \max\{j : x_j | w_i\}$, $1 \leq i \leq t$.

Proof. We claim that

$$|S_1 V| = |P_{d+1}| + \sum_{i=1}^t |S_1(w_i)| - \sum_{i=1}^t |S_1(w_i) \cap P_{d+1}| - \sum_{i=2}^t |S_1(w_i) \cap S_1(w_1, \dots, w_{i-1})|.$$

We prove the claim by induction on t . If $t = 1$, then

$$|S_1 V| = |P_{d+1}| + |S_1(w_1)| - |S_1(w_1) \cap P_{d+1}|.$$

Let $t > 1$ and $V_1 = P_d + (w_1, \dots, w_{t-1})$. By the inductive step we obtain that $|S_1 V|$ is equal to

$$|P_{d+1}| + \sum_{i=1}^t |S_1(w_i)| - \sum_{i=1}^{t-1} |S_1(w_i) \cap P_{d+1}| - \sum_{i=2}^{t-1} |S_1(w_i) \cap S_1(w_1, \dots, w_{i-1})| - |S_1(w_t) \cap S_1 \overline{V}|.$$

By Lemma 3.7, we have $|S_1(w_t) \cap S_1 V_1| = |S_1(w_t) \cap P_{d+1}| + |S_1(w_t) \cap S_1(w_1, \dots, w_{t-1})|$. We proved the claim.

Let $2 \leq j \leq t$. If $i < m(w_j)$ such that $x_i \nmid w_j$, then $x_i w_j \in S_1(w_1, \dots, w_{j-1})$. So $|S_1(w_i) \cap S_1(w_1, \dots, w_{i-1})| = m(w_j) - d$. Therefore

$$\begin{aligned} |S_1 V| &= |S_{d+1}| - \binom{n}{d+1} + tn - td - \sum_{i=2}^t (m(w_i) - d) \\ &= \binom{d+n}{d+1} - \binom{n}{d+1} + tn - td - \sum_{i=2}^t (m(w_i) - d) \\ &= \binom{d+n}{d+1} - \binom{n}{d+1} + tn - td - \sum_{i=1}^t (m(w_i) - d) \\ &= \binom{d+n}{d+1} - \binom{n}{d+1} + \sum_{i=1}^t (n - m(w_i)). \end{aligned}$$

□

4. THE MAIN RESULT

In this section we prove that the EGH Conjecture is true if f_i splits into linear factors for all i . We begin with the following lemma.

Lemma 4.1. *Let $P = \langle f_1, \dots, f_n \rangle$ be an ideal of S generated by a regular sequence with $\deg(f_i) = a_i$ and $n \geq 2$. Assume that $f_n = q_1 \cdots q_s$, where $q_1, \dots, q_s \in S_1$. Then*

- (a) $H(S/P + \langle q_m \rangle) = H(S/P + \langle q_k \rangle)$ for all $1 \leq m, k \leq s$.
- (b) $H(S/(P : q_1 \cdots q_j) + \langle q_m \rangle) = H(S/(P : q_1 \cdots q_j) + \langle q_k \rangle)$ for all $1 \leq j \leq s-1$ and $j < m, k \leq s$.

Proof. First, we will prove (a). Let $1 \leq m, k \leq s$. Note that $P + \langle q_m \rangle / \langle q_m \rangle$ and $P + \langle q_k \rangle / \langle q_k \rangle$ are ideals in $S/\langle q_m \rangle$ and $S/\langle q_k \rangle$, respectively, generated by $\overline{f_1}, \dots, \overline{f_{n-1}}$. Note also that $\overline{f_1}, \dots, \overline{f_{n-1}}, q_m$ and $\overline{f_1}, \dots, \overline{f_{n-1}}, q_k$ are regular sequences. By part (c) of Lemma 2.7, we obtain that $\overline{f_1}, \dots, \overline{f_{n-1}}$ is a regular sequence in $S/\langle q_m \rangle$ and $S/\langle q_k \rangle$. By part (a) of Lemma 2.7, we obtain that $H(S/P + \langle q_m \rangle) = H(S/P + \langle q_k \rangle)$.

Now, we prove (b). Let $1 \leq j \leq s-1$ and $j < m, k \leq s$. Assume that

$$h = h_1 + h_2 \in (P : q_1 \cdots q_j) + \langle q_m \rangle,$$

where $h_1 \in (P : q_1 \cdots q_j)$ and $h_2 \in \langle q_m \rangle$. Since $h_1 q_1 \cdots q_j \in P$, it follows that $h_1 q_1 \cdots q_j = g_1 f_1 + \cdots + g_n f_n$, where $g_1, \dots, g_n \in S$; i.e.,

$$g_1 f_1 + \cdots + g_{n-1} f_{n-1} + q_1 \cdots q_j (g_n q_{j+1} \cdots q_s - h_1) = 0.$$

Since $f_1, \dots, f_{n-1}, q_1 \cdots q_j$ is a regular sequence, it follows that $g_n q_{j+1} \cdots q_s - h_1 \in \langle f_1, \dots, f_{n-1} \rangle$. So $\overline{h_1} \in \langle \overline{f_1}, \dots, \overline{f_{n-1}} \rangle$ in the ring $S/\langle q_m \rangle$, which implies that $\overline{h} \in \langle \overline{f_1}, \dots, \overline{f_{n-1}} \rangle$. Conversely, $\overline{f_i} \in (P : q_1 \cdots q_j) + \langle q_m \rangle / \langle q_m \rangle$ for all $1 \leq i \leq n-1$. So $(P : q_1 \cdots q_j) + \langle q_m \rangle / \langle q_m \rangle$ is an ideal in $S/\langle q_m \rangle$ generated by $\overline{f_1}, \dots, \overline{f_{n-1}}$. Similarly, $(P : q_1 \cdots q_j) + \langle q_k \rangle / \langle q_k \rangle$ is an ideal in $S/\langle q_k \rangle$ generated by $\overline{f_1}, \dots, \overline{f_{n-1}}$. By Lemma 2.7, it follows that $H(S/(P : q_1 \cdots q_j) + \langle q_k \rangle) = H(S/(P : q_1 \cdots q_j) + \langle q_m \rangle)$. □

Theorem 4.2. *Assume that the EGH Conjecture holds in $k[x_1, \dots, x_{n-1}]$, where $n \geq 2$. If I is a graded ideal in $S = k[x_1, \dots, x_n]$ containing a regular sequence $f_1, \dots, f_{n-1}, f_n = q_1 \cdots q_s$ of degrees $\deg(f_i) = a_i$ such that $q_i \in S_1$ for all $1 \leq i \leq s$, then I has the same Hilbert function as a graded ideal in S containing $x_1^{a_1}, \dots, x_n^{a_n}$.*

Proof. We check the property (b) of Lemma 2.6. Let $d \geq 0$. We need to find a graded ideal K in S containing $x_1^{a_1}, \dots, x_n^{a_n}$ such that $H(S/I, d) = H(S/K, d)$ and $H(S/I, d+1) \leq H(S/K, d+1)$. Let J to be the ideal generated by f_1, \dots, f_n and I_d . By renaming the linear polynomials q_1, \dots, q_s , we may assume without loss of generality that

$$\begin{aligned} |J_d \cap \langle q_1 \rangle_d| &\geq |J_d \cap \langle q_i \rangle_d| \text{ for all } 2 \leq i \leq s, \\ |(J : q_1)_{d-1} \cap \langle q_2 \rangle_{d-1}| &\geq |(J : q_1)_{d-1} \cap \langle q_i \rangle_{d-1}| \text{ for all } 3 \leq i \leq s, \\ |(J : q_1 q_2)_{d-2} \cap \langle q_3 \rangle_{d-2}| &\geq |(J : q_1 q_2)_{d-2} \cap \langle q_i \rangle_{d-2}| \text{ for all } 4 \leq i \leq s, \\ &\vdots \\ |(J : q_1 \cdots q_{s-2})_{d-(s-2)} \cap \langle q_{s-1} \rangle_{d-(s-2)}| &\geq |(J : q_1 \cdots q_{s-2})_{d-(s-2)} \cap \langle q_s \rangle_{d-(s-2)}|. \end{aligned}$$

By considering the short exact sequences

$$\begin{aligned}
0 \rightarrow S/(J : q_1) &\xrightarrow{g \mapsto gq_1} S/J \xrightarrow{g \mapsto g} S/J + \langle q_1 \rangle \rightarrow 0, \\
0 \rightarrow S/(J : q_1 q_2) &\xrightarrow{g \mapsto gq_2} S/(J : q_1) \xrightarrow{g \mapsto g} S/(J : q_1) + \langle q_2 \rangle \rightarrow 0, \\
0 \rightarrow S/(J : q_1 q_2 q_3) &\xrightarrow{g \mapsto gq_3} S/(J : q_1 q_2) \xrightarrow{g \mapsto g} S/(J : q_1 q_2) + \langle q_3 \rangle \rightarrow 0, \\
&\vdots \\
0 \rightarrow S/(J : q_1 \cdots q_{s-1}) &\xrightarrow{g \mapsto gq_{s-1}} S/(J : q_1 \cdots q_{s-2}) \xrightarrow{g \mapsto g} S/(J : q_1 \cdots q_{s-2}) + \langle q_{s-1} \rangle \rightarrow 0.
\end{aligned}$$

we see that $H(S/J, t)$ is equal to

$$H(S/J + \langle q_1 \rangle, t) + \sum_{i=1}^{s-2} H(S/(J : q_1 \cdots q_i) + \langle q_{i+1} \rangle, t-i) + H(S/(J : q_1 \cdots q_{s-1}), t-(s-1))$$

for all $t \geq 0$. Let $J_0 = J + \langle q_1 \rangle$, $J_{s-1} = (J : q_1 \cdots q_{s-1})$, and for $1 \leq i \leq s-2$ let $J_i = (J : q_1 \cdots q_i) + \langle q_{i+1} \rangle$. Note that $q_{i+1} \in J_i$ and $H(\frac{S/\langle q_{i+1} \rangle}{J_i/\langle q_{i+1} \rangle}) = H(S/J_i)$ for all $0 \leq i \leq s-1$. Set $\overline{S} = k[x_1, \dots, x_{n-1}]$. For all $0 \leq i \leq s-1$, $S/\langle q_{i+1} \rangle$ is isomorphic to \overline{S} , so by the hypothesis there is an ideal in \overline{S} containing $x_1^{a_1}, \dots, x_{n-1}^{a_{n-1}}$ with the same Hilbert function as J_i . For all $0 \leq i \leq s-1$, let L_i be the lex-plus-powers ideal in \overline{S} containing $x_1^{a_1}, \dots, x_{n-1}^{a_{n-1}}$ such that $H(\overline{S}/L_i) = H(S/J_i)$.

Claim: $L_{i,j} \subseteq L_{i+1,j}$ for all $0 \leq i \leq s-2$ and $j \leq d-i$, where $L_{i,j}$ is the j -th component of the ideal L_i .

Proof of the claim: Assume that $i = 0$. If $j < d$, then by part (a) of Lemma 4.1 we obtain

$$|J_{0,j}| = |J_j + \langle q_1 \rangle_j| = |P_j + \langle q_1 \rangle_j| = |P_j + \langle q_2 \rangle_j| \leq |J_{1,j}|.$$

If $j = d$, then by our assumption we obtain

$$\begin{aligned}
|J_{0,d}| &= |J_d| + |\langle q_1 \rangle_d| - |J_d \cap \langle q_1 \rangle_d| \\
&\leq |J_d| + |\langle q_1 \rangle_d| - |J_d \cap \langle q_2 \rangle_d| \\
&= |J_d| + |\langle q_2 \rangle_d| - |J_d \cap \langle q_2 \rangle_d| \\
&= |J_d + \langle q_2 \rangle_d| \\
&\leq |J_{1,d}|.
\end{aligned}$$

This means that $H(S/J_0, j) \geq H(S/J_1, j)$ for all $j \leq d$. So $H(\overline{S}/L_0, j) \geq H(\overline{S}/L_1, j)$ for all $j \leq d$. Since L_0 and L_1 are lex-plus-powers ideals, it follows that $L_{0,j} \subseteq L_{1,j}$ for all $j \leq d$.

Let $0 < i \leq s-2$. If $j < d-i$, then by part (b) of Lemma 4.1 we obtain

$$|J_{i,j}| = |(J : q_1 \cdots q_i)_j + \langle q_{i+1} \rangle_j| = |(P : q_1 \cdots q_i)_j + \langle q_{i+1} \rangle_j| = |(P : q_1 \cdots q_i)_j + \langle q_{i+2} \rangle_j| \leq |J_{i+1,j}|.$$

If $j = d-i$, then by our assumption we obtain

$$\begin{aligned}
|J_{i,d-i}| &= |(J : q_1 \cdots q_i)_{d-i}| + |\langle q_{i+1} \rangle_{d-i}| - |(J : q_1 \cdots q_i)_{d-i} \cap \langle q_{i+1} \rangle_{d-i}| \\
&\leq |(J : q_1 \cdots q_i)_{d-i}| + |\langle q_{i+1} \rangle_{d-i}| - |(J : q_1 \cdots q_i)_{d-i} \cap \langle q_{i+2} \rangle_{d-i}| \\
&= |(J : q_1 \cdots q_i)_{d-i}| + |\langle q_{i+2} \rangle_{d-i}| - |(J : q_1 \cdots q_i)_{d-i} \cap \langle q_{i+2} \rangle_{d-i}| \\
&= |(J : q_1 \cdots q_i)_{d-i} + \langle q_{i+2} \rangle_{d-i}| \\
&\leq |J_{i+1,d-i}|.
\end{aligned}$$

Similarly, we conclude that $L_{i,j} \subseteq L_{i+1,j}$ for all $j \leq d-i$, and proving the claim.

Let $K_s = \{zx_n^{s+j} : z \in \text{Mon}(\overline{S}) \wedge j \geq 0\}$ and $K_i = \{zx_n^i : z \in \text{Mon}(L_i)\}$ for all $0 \leq i \leq s-1$. Define K to be the ideal generated by $\bigcup_{0 \leq i \leq s} K_i$. Since $x_n^s \in K_s$ and $x_i^{a_i} \in K_0$ for all $1 \leq i \leq n-1$, it follows that $x_1^{a_1}, \dots, x_n^{a_n} \in K$.

Claim: If w is a monomial in K of degree t , where $0 \leq t \leq d+1$, then $w \in \bigcup_{0 \leq i \leq s} K_i$.

Proof of the claim: There exists a monomial u in $\bigcup_{0 \leq i \leq s} K_i$ such that $u|w$; i.e., $w = vu$ for some monomial $v \in S$. If $u \in K_s$, then $w \in K_s$. Assume that $u = zx_n^i \in K_i$, where $z \in L_i$ for some $0 \leq i \leq s-1$. If $x_n \nmid v$, then $w \in \bigcup_{0 \leq i \leq s} K_i$. Assume that $x_n|v$. Let $r = \max\{j : x_n^j|v\}$. If $i+r \geq s$, then $w \in K_s$. So we may assume that $i+r < s$. By the previous claim, we obtain that $L_{i,j} \subseteq L_{i+r,j}$ for all $j \leq d-(i+r-1)$. Since $\deg(z) \leq d+1-(i+r)$, it follows that $z \in L_{i+r}$. So $\frac{v}{x_n^r}z \in L_{i+r}$, and then $\frac{v}{x_n^r}zx_n^{r+i} = w \in K_{i+r}$. Hence, we proved the claim.

We conclude that the number of monomials in K of degree t , where $0 \leq t \leq d+1$, is equal to $\sum_{i=0}^{s-1} |L_{i,t-i}| + \sum_{i=0}^{t-s} |\overline{S}_i|$. Since $|S_t| = \sum_{0 \leq i \leq t} |\overline{S}_i|$, it follows that

$$|S_t| - |K_t| = \sum_{i=t-(s-1)}^t |\overline{S}_i| - \sum_{i=0}^{s-1} |L_{i,t-i}| = \sum_{i=0}^{s-1} |\overline{S}_{t-i}| - \sum_{i=0}^{s-1} |L_{i,t-i}|.$$

So $H(S/K, t) = \sum_{i=0}^{s-1} H(\overline{S}/L_i, t-i) = \sum_{i=0}^{s-1} H(S/J_i, t-i) = H(S/J, t)$. In particular,

$$\begin{aligned} H(S/K, d) &= H(S/J, d) = H(S/I, d) \text{ and} \\ H(S/K, d+1) &= H(S/J, d+1) \geq H(S/I, d+1). \end{aligned}$$

□

Corollary 4.3. *If I is a graded ideal in S containing a regular sequence f_1, \dots, f_n with $\deg(f_i) = a_i$ such that f_i splits into linear factors for all i , then I has the same Hilbert function as a graded ideal in S containing $x_1^{a_1}, \dots, x_n^{a_n}$.*

Since the EGH Conjecture holds when $n = 2$, we obtain the following.

Corollary 4.4. *Let $n \geq 3$. If I is a graded ideal in S containing a regular sequence f_1, \dots, f_n with $\deg(f_i) = a_i$ such that f_i splits into linear factors for all $3 \leq i \leq n$, then I has the same Hilbert function as a graded ideal in S containing $x_1^{a_1}, \dots, x_n^{a_n}$.*

By 4.3, the EGH Conjecture is equivalent to the following conjecture.

Conjecture 4.5. *If I is a homogeneous ideal in S containing a regular sequence f_1, \dots, f_n of degrees $\deg(f_i) = a_i$, then I has the same Hilbert function as an ideal containing a regular sequence g_1, \dots, g_n of degrees $\deg(g_i) = a_i$, where g_i splits into linear factors for all i .*

Example 4.6. Let $S = \mathbb{C}[x_1, \dots, x_5]$, $f_i = x_i(\sum_{j=1}^{i-1} x_j) + x_i(\sum_{j=i}^5 x_j)$ for all $1 \leq i \leq 5$ and

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 1 \end{pmatrix}.$$

Since $\det A[i_1, \dots, i_r] \neq 0$ for all $1 \leq r \leq 5$ and $1 \leq i_1 < \dots < i_r \leq 5$, it follows that f_1, \dots, f_5 is a regular sequence in S . Assume that $I = \langle f_1, \dots, f_5, x_1x_2 + x_1x_3, x_1^2 + x_4x_5 \rangle$. In this example, we construct an ideal in S with the same Hilbert function

as I , using the Hilbert functions of $J_0 = I + \langle x_5 \rangle$ and $J_1 = (I : x_5)$. Computation with Macaulay2 shows that

$$H_{S/I} = (1, 5, 8, 3, 0, 0, \dots), H_{S/J_0} = (1, 4, 4, 1, 0, 0, \dots) \text{ and } H_{S/J_1} = (1, 4, 2, 0, 0, \dots)$$

are the Hilbert sequence of I , J_0 and J_1 , respectively. Denote by R the polynomial ring $\mathbb{C}[x_1, \dots, x_4]$. Let

$$L_0 = \langle x_1^2, \dots, x_4^2, x_1x_2, x_1x_3 \rangle \subset R \text{ and } L_1 = \langle x_1^2, \dots, x_4^2, x_1x_2, x_1x_3, x_1x_4, x_2x_3 \rangle \subset R.$$

Note that L_0 and L_1 are lex-plus-powers ideals in R . We can see that $L_{0,0} = L_{0,1} = (0)$ and

$$\begin{aligned} L_{0,2} &= (x_1^2, x_1x_2, x_1x_3, x_2^2, x_3^2, x_4^2), \\ L_{0,3} &= (w : w \in \text{Mon}(R_3) \text{ and } w \neq x_2x_3x_4), \\ L_{0,j} &= R_j \text{ for all } j \geq 4. \end{aligned}$$

So we have $H_{R/L_0} = H_{S/J_0}$. Also, we have $L_{1,0} = L_{1,1} = (0)$ and

$$\begin{aligned} L_{1,2} &= (x_1^2, x_1x_2, x_1x_3, x_1x_4, x_2^2, x_2x_3, x_3^2, x_4^2), \\ L_{1,j} &= R_j \text{ for all } j \geq 3. \end{aligned}$$

So we have $H_{R/L_1} = H_{S/J_1}$. Let K to be the ideal in S generated by

$$\text{Mon}(L_0) \cup \{wx_5 : w \in \text{Mon}(L_1)\} \cup \{x_5^2\}.$$

Then $K = \langle x_1^2, x_2^2, x_3^2, x_4^2, x_5^2, x_1x_2, x_1x_3, x_1x_4x_5, x_2x_3x_5 \rangle$. It is clear that $|S_0/K_0| = 1$ and $|S_1/K_1| = 5$. Since $S_2/K_2 = (\overline{x_1x_4}, \overline{x_1x_5}, \overline{x_2x_3}, \overline{x_2x_4}, \overline{x_2x_5}, \overline{x_3x_4}, \overline{x_3x_5}, \overline{x_4x_5})$, it follows that $|S_3/K_3| = 8$. Also we have $S_3/K_3 = (\overline{x_2x_3x_4}, \overline{x_2x_4x_5}, \overline{x_3x_4x_5})$ and $K_j = S_j$ for all $j \geq 4$. Thus

$$H_{S/K} = (1, 5, 8, 3, 0, 0, \dots) = H_{S/I}.$$

Example 4.7. Let $S = \mathbb{C}[x_1, \dots, x_6]$, $f_i = x_i(\sum_{j=1}^{i-1} -x_j) + x_i(\sum_{j=i}^6 x_j)$ for all $1 \leq i \leq 5$ and $f_6 = x_6^2(-x_1 - x_2 - x_3 - x_4 - x_5 + x_6)$. Since $f_1, \dots, f_5, \frac{f_6}{x_6}$ is a regular sequence, it follows that f_1, \dots, f_6 is a regular sequence in S . Assume that

$$I = \langle f_1, \dots, f_6, x_1x_2 + x_3x_4, x_1x_6 + x_5^2, x_2^2x_3 \rangle.$$

Computation with Macaulay2 shows that

$$\begin{aligned} H_{S/I} &= (1, 6, 14, 13, 2, 0, \dots), \\ H_{S/I+\langle x_6 \rangle} &= (1, 5, 8, 2, 0, \dots), \\ H_{S/(I:x_6)+\langle x_6 \rangle} &= (1, 5, 6, 0, \dots), \\ H_{S/(I:x_6^2)} &= (1, 5, 2, 0, \dots). \end{aligned}$$

Also we have

$$|I_2 \cap \langle x_6 \rangle_2| = |I_2 \cap \langle -x_1 - x_2 - x_3 - x_4 - x_5 + x_6 \rangle_2|$$

and

$$|(I : x_6)_1 \cap \langle x_6 \rangle_1| = |(I : x_6)_1 \cap \langle -x_1 - x_2 - x_3 - x_4 - x_5 + x_6 \rangle_1|.$$

We construct an ideal in S with the same Hilbert function as I , using the Hilbert functions of $I + \langle x_6 \rangle$, $(I : x_6) + \langle x_6 \rangle$ and $(I : x_6^2)$. Denote by J_0 , J_1 and J_2 the ideals $I + \langle x_6 \rangle$, $(I : x_6) + \langle x_6 \rangle$ and $(I : x_6^2)$, respectively. Let $R = \mathbb{C}[x_1, \dots, x_5]$ and $L_0 = \langle x_1^2, \dots, x_5^2, x_1x_2, x_1x_3, x_1x_4x_5, x_2x_3x_4, x_2x_3x_5 \rangle \subset R$. An easy calculation shows that L_0 is a lex-plus-powers ideal in R and $H_{R/L_0} = (1, 5, 8, 0, \dots) = H_{S/I+\langle x_6 \rangle}$. Let $L_1 = \langle x_1^2, \dots, x_5^2, x_1x_2, x_1x_3, x_1x_4, x_1x_5, x_2x_3x_4, x_2x_3x_5, x_2x_4x_5, x_3x_4x_5 \rangle \subset R$. We can see that L_1 is a lex-plus-powers ideal and $H_{R/L_1} = (1, 5, 6, 0, \dots) = H_{S/J_1}$. Let

$L_2 = \langle x_1^2, \dots, x_5^2, x_1x_2, x_1x_3, x_1x_4, x_1x_5, x_2x_3, x_2x_4, x_2x_5, x_3x_4 \rangle \subset R$. Also we have that L_2 is a lex-plus-powers ideal in R and $H_{R/L_2} = (1, 5, 3, 0, \dots) = H_{S/J_2}$. Let K to be the ideal in S generated by $\text{Mon}(L_0) \cup \{wx_6 : w \in \text{Mon}(L_1)\} \cup \{wx_6^2 : w \in \text{Mon}(L_2)\} \cup \{x_6^3\}$. The ideal K generated by

$$\{x_1^2, \dots, x_5^2, x_6^3, x_1x_2, x_1x_3, x_1x_4x_5, x_2x_3x_4, x_2x_3x_5, x_1x_4x_6\}$$

$$\cup$$

$$\{x_1x_5x_6, x_2x_4x_5x_6, x_3x_4x_5x_6, x_2x_3x_6^2, x_2x_4x_6^2, x_2x_5x_6^2, x_3x_4x_6^2\}.$$

Computation with Macaulay2 shows that $H_{S/K} = (1, 6, 14, 13, 2, 0, \dots) = H_{S/I}$.

REFERENCES

- [1] A. Abedelfatah. Macaulay-Lex rings. *Journal of Algebra*, 374:122-131, 2013.
- [2] A. Aramova, J. Herzog, and T. Hibi. Gotzmann Theorems for Exterior Algebras and Combinatorics. *Journal of Algebra*, 191(1):174-211, 1997.
- [3] G. Caviglia and D. Maclagan. Some cases of the Eisenbud-Green-Harris conjecture. *Mathematical Research Letters*, 15(3):427-433, 2008.
- [4] R.I.X. Chen. Some special cases of the Eisenbud-Green-Harris Conjecture.
- [5] G.F. Clements and B. Lindström. A generalization of a combinatorial theorem of Macaulay. *Journal of Combinatorial Theory*, 7(3):230-238, 1969.
- [6] S.M. Cooper. Growth conditions for a family of ideals containing regular sequences. *Journal of Pure and Applied Algebra*, 212(1):122-131, 2008.
- [7] S.M. Cooper. The Eisenbud-Green-Harris Conjecture for Ideals of Points. 2008.
- [8] D. Eisenbud, M. Green, and J. Harris. Higher Castelnuovo Theory. *Astérisque*, 218:187, 1993.
- [9] J. Herzog and T. Hibi. *Monomial Ideals*, volume 260. Springer Verlag, 2010.
- [10] J. Herzog and D. Popescu. Hilbert functions and generic forms. *Compositio Mathematica*, 113(1):1-22, 1998.
- [11] G.O.H. Katona. A theorem of finite sets. *Theory of graphs*, pages 187-207, 1968.
- [12] J.B. Kruskal. The Number of Simplices in a Complex. *Mathematical optimization techniques*, page 251, 1963.
- [13] F. Macaulay. Some properties of enumeration in the theory of modular systems. *Proceedings of the London Mathematical Society*, 2(1):531, 1927.
- [14] H. Matsumura. *Commutative ring theory*, volume 8 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1989.
- [15] J. Mermin and I. Peeva. Lexifying ideals. *Mathematical Research Letters*, 13(2/3):409, 2006.
- [16] B.P. Richert. A study of the lex plus powers conjecture. *Journal of Pure and Applied Algebra*, 186(2):169-183, 2004.
- [17] D.A. Shakin. Piecewise lexsegment ideals. *Sbornik: Mathematics*, 194:1701, 2003.

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